

Equivalent condition for approximately Cohen-Macaulay complexes

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ABSTRACT. We give a necessary and sufficient condition for a simplicial complex to be approximately Cohen-Macaulay. Namely it is approximately Cohen-Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear and generated in two consecutive degrees. This completes the result of Herzog and Hibi who proved that a simplicial complex is sequentially Cohen-Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear.

1. Introduction

In [5] Eagon and Reiner proved that a simplicial complex is Cohen-Macaulay if and only if the ideal associated to its Alexander dual has linear resolution. Later Herzog and Hibi [8] generalized it and proved that a simplicial complex is sequentially Cohen-Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear. We use their result to give a similar equivalent condition for a simplicial complex to be approximately Cohen-Macaulay.

We begin with a brief introduction to the topic. When we say that a simplicial complex is Cohen-Macaulay, sequentially Cohen-Macaulay, or approximately Cohen-Macaulay, we always think that its Stanley-Reisner ring has this property.

DEFINITION 1. *For a simplicial complex Δ on the set of vertices $\{1, \dots, n\}$ and a field \mathbb{K} , the Stanley-Reisner ring (or face ring) is the ring $\mathbb{K}[x_1, \dots, x_n]/I_\Delta = \mathbb{K}[\Delta]$, where I_Δ is generated by all squarefree monomials $x_{i_1} \cdots x_{i_l}$ for which $\{i_1, \dots, i_l\}$ is not a face in Δ .*

We recall combinatorial description of Cohen-Macaulay complexes:

DEFINITION 2. *Let σ be a simplex in a simplicial complex Δ . The link of σ in Δ , denoted by $lk_\Delta \sigma$, is the simplicial complex $\{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$.*

THEOREM 1. (Reisner [10]) *Let $R = \mathbb{K}[\Delta]$ be the face ring of Δ . Then the following conditions are equivalent:*

- (1) *R is Cohen-Macaulay ring.*
- (2) *$\tilde{H}_i(lk_\Delta \sigma) = 0$ if $i < \dim(lk_\Delta \sigma)$ for all simplices $\sigma \in \Delta$.*

Key words and phrases. approximately Cohen-Macaulay ring, Stanley-Reisner ring, Alexander dual complex, sequentially Cohen-Macaulay ring.

For some techniques of counting homology we refer the reader to Section 3.2 of [9]. We need also the following definitions:

DEFINITION 3. [7] *A non Cohen-Macaulay local ring A is called approximately Cohen-Macaulay if there is an element a in the maximal ideal such that $A/(a^n)$ is Cohen-Macaulay ring of dimension $\dim(A) - 1$ for all $n > 0$.*

DEFINITION 4. *A ring A of dimension d is called sequentially Cohen-Macaulay if there exists a filtration of ideals of A :*

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = A$$

such that each D_i/D_{i-1} is Cohen-Macaulay and

$$0 < \dim(D_1/D_0) < \dim(D_2/D_1) < \cdots < \dim(D_t/D_{t-1}) = d.$$

DEFINITION 5. *Let Δ be a simplicial complex on the set of vertices V , we define its Alexander dual to be $\Delta^* = \{V \setminus \sigma : \sigma \notin \Delta\}$.*

DEFINITION 6. *We say that a graded ideal $I \subset A$ is componentwise linear if I_j has linear resolutions for each degree j .*

There is a nice description of approximately Cohen-Macaulay rings:

PROPOSITION 1. [3] *Let A be a non Cohen-Macaulay local ring of dimension d . Then the following conditions are equivalent:*

- (1) *A is an approximately Cohen-Macaulay ring.*
- (2) *A is a sequentially Cohen-Macaulay ring with filtration $0 = D_0 \subset D_1 \subset D_2 = A$, where $\dim(D_1) = d - 1$.*

2. Equivalent condition

We will make use of the following result of Herzog and Hibi [8].

THEOREM 2. [8] *Let Δ be a simplicial complex. Then Stanley-Reisner ring $\mathbb{K}[\Delta]$ is sequentially Cohen-Macaulay if and only if I_{Δ^*} , the ideal associated to its Alexander dual, is componentwise linear.*

Our theorem reads as follows.

THEOREM 3. *Let Δ be a simplicial complex. Then the Stanley-Reisner ring $\mathbb{K}[\Delta]$ is approximately Cohen-Macaulay if and only if I_{Δ^*} , ideal associated to its Alexander dual, is componentwise linear and generated in two consecutive degrees.*

PROOF. By Proposition 1, $\mathbb{K}[\Delta]$ is approximately Cohen-Macaulay if and only if $\mathbb{K}[\Delta]$ is a sequentially Cohen-Macaulay ring with filtration

$$0 = D_0 \subset D_1 \subset D_2 = \mathbb{K}[\Delta],$$

where $\dim(D_1) = d - 1$. Due to the Theorem 2 of Herzog and Hibi this is equivalent to componentwise linearity of I_{Δ^*} , and existence of a filtration

$$0 = D_0 \subset D_1 \subset D_2 = \mathbb{K}[\Delta],$$

where $\dim(D_1) = d - 1$. From Appendix of [1] we get that if such a filtration exists, then it is unique and coincides with the one given by

$$0 = M_0 \subset \cdots \subset M_{i-1} = I_{\Delta, \Delta^{(j_i-1)}} \subset \cdots \subset \mathbb{K}[\Delta],$$

where $I_{\Delta, \Delta^{(j_i-1)}}$ is the ideal in $\mathbb{K}[\Delta]$ generated by monomials x_A , with $A \in \Delta \setminus \Delta^{(j_i-1)}$. The simplicial complex $\Delta^{(j_i-1)}$ is generated by faces of Δ of dimension at least $j_i - 1$, where $j_1 - 1 < \dots < j_s - 1$ are the dimensions of facets of Δ . We have also that $\dim(\Delta^{(j_i-1)}) = j_i - 1$. Hence the desired filtration exists if and only if Δ has facets of dimension d and $d - 1$. Ideal I_{Δ^*} is generated by monomials x_A for $A \notin \Delta^*$, that is, for $A = V \setminus \sigma$, where $\sigma \in \Delta$. We have to take all x_A corresponding to facets and they all already generate ideal. Hence the ideal I_{Δ^*} is generated in two consecutive degrees $v - d$ and $v - (d - 1)$, where $|V| = v$. Since each step of our reasoning was an equivalence, the contrary also holds. \square

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